# A generalization of Aluthge transformation using semi-hyponormal operators

**C.V.Seshaiah**[1] **and K.Meenambika**[2]

[1] **Department of Mathematics, Sri Ramakrishna Engineering College, Coimbatore-641022**

[2] **Department of Mathematics, Sengunthar Engineering College, Tiruchengode-637205**

**Email: cvseshaiah@gmail.com**[1] **, meenabalaji08@gmail.com**[2]

**Abstract**

In this paper, different properties of Aluthge transform are defined more generally for any s and t such as $s \geq 0$ and $t \geq 0$the Aluthge transformation of an operator B are studied using semi-hyponormal operators.

**Keywords :** Aluthge transform, hyponormal operators, semi-hyponormal operators, class A operator, quasiclass A operator, quasiclass(A,m) operator , posinormal operator, quasiposinormal operator.

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## 1. Introduction

In [1] A.Aluthge introduced the operator B with its polar decomposition $B = \bigcup |B| = |B^*| \bigcup$.In[3] Daoxing Xia worked on semi-hyponormal n-tuple of operators. In [4] A. Uchiyama introduced

Weyl's Theorem for class A operator. In[6] H. Crawford Rhaly have worked on Posinormal operator.In[7] I.H.Kim have introduced and studied On (p,k) quasihyponormal operators.In[8] ] T. Furuta , M. Ito and T. Yamazaki have studied A sub class of paranormal operators including class of log- hyponormal and several classes. In this paper we are interested in some of the properties of Aluthge transformation using semi- hyponormal operators.

## 2.Preliminaries

Let B be a bounded linear operator on a Hilbert space H. Throughout our discussion, by an operator, we shall mean a bounded linear transformation on a Hilbert space H.

## 3.Definitions

**Definition 3.1** :

A generalization of a normal operator is called a Hyponormal operator . A bounded linear operator B on a Hilbert space H is said to be p-hyponormal if $\left(B^*B\right)^p \geq \left(BB^*\right)^p$ .If p=1, then B is called a hyponormal operator.

**Definition 3.2 :**

If p=1/2,then B is called a semi-hyponormal operator.

(i.e) $\left(B^*B\right)^{\frac{1}{2}} \geq \left(BB^*\right)^{\frac{1}{2}}$

**Definition 3.3 :**

An operator B is said to be in class A iff $\left(B^*\left|B\right|^2 B\right)^{\frac{1}{2}} \geq B^*B$

**Definition 3.4 :**

An operator B belongs to quasiclass A if

$$B^{*}\left(\left|B^{2}\right|-\left|B\right|^{2}\right)B\geq 0$$

**Definition 3.5:**

An operator B in B(H) is called a m-quasiclass A operator for a positive integer m if

$$B^{*^{m}}\left(\left|B^{2}\right|-\left|B\right|^{2}\right)B^{m}\geq 0$$

**Definition 3.6:**

An operator B is quasiposinormal if,

$$\left(BB^{*}\right)^{2}\leq c^{2}B^{*^{2}}B^{2}$$

## 4. Posinormal operators

**Theorem 4.1:**

If B is a semi-hyponormal operator, then for any (s,t),B(s,t) is posinormal.

**Proof:**

Given B is semi-hyponormal,then

$$\left|B^{*}B\right|^{\frac{1}{2}}\geq\left|BB^{*}\right|^{\frac{1}{2}} \quad \text{(or)}$$
$$BB^{*}\leq c^{2}B^{*}B$$

$$\left|B\right|\geq\left|B^{*}\right|$$

B is posinormal if $BB^* - c^2 B^* B \le 0$ for some c>0

Now,

$$\Rightarrow B(s,t)B^*(s,t) - c^2 B^*(s,t)B(s,t) \le 0$$

$$LHS = B(s,t)B^*(s,t) - c^2 B^*(s,t)B(s,t)$$

$$= |B|^s \cup |B|^t \left(|B|^s \cup |B|^t\right)^* - c^2 \left[\left(|B|^s \cup |B|^t\right)^* \left(|B|^s \cup |B|^t\right)\right]$$

$$= |B|^s \cup |B|^t |B|^t \cup^* |B|^s - c^2 \left[|B|^t \cup^* |B|^s |B|^s \cup |B|^t\right]$$

$$= |B|^s \cup |B|^{2t} \cup^* |B|^s - c^2 \left[|B|^t \cup^* |B|^{2s} \cup |B|^t\right]$$

$$= |B|^s |B^*|^{2t} |B|^s - c^2 |B|^t |B^*|^{2s} |B|^t$$

$$\le |B|^s |B|^{2t} |B|^s - c^2 |B|^t |B|^{2s} |B|^t$$

$$\le |B|^{2(s+t)} - c^2 |B|^{2(s+t)}$$

$$= \left(1 - c^2\right)|B|^{2(s+t)}$$

$$\le 0, c > 0$$

=RHS

$$BB^* - c^2 B^* B \le 0$$

$\Rightarrow$B(s,t) is posinormal.

**Theorem 4.2:**

If B is semi- hyponormal, then for any (s,t), B(s,t) is quasiposinormal.

**Proof:**

Let us assume that B is quasiposinormal then,

$$\left(B^*\right)^2 \leq c^2 B^{*^2} B^2$$

$$\left(BB^*\right)^2 - c^2\left(B^{*^2} B^2\right) \leq 0$$

$$\Rightarrow \left[B(s,t)B^*(s,t)\right]^2 - c^2\left[B^{*^2}(s,t)B^2(s,t)\right] \leq 0$$

$$LHS = \left[B(s,t)B^*(s,t)\right]^2 - c^2\left[B^*(s,t)B(s,t)\right]^2$$

$$= \left[|B|^s \cup |B|^t |B|^t \cup^* |B|^s\right]^2 - c^2\left[|B|^t \cup^* |B|^s |B|^s \cup |B|^t\right]^2$$

$$= \left[|B|^s \cup |B|^{2t} \cup^* |B|^s\right]^2 - c^2\left[|B|^t \cup^* |B|^{2s} \cup |B|^t\right]^2$$

$$= \left[|B|^s \left|B^*\right|^{2t} |B|^s\right]^2 - c^2\left[|B|^t \left|B^*\right|^{2s} |B|^t\right]^2$$

$$= \left[|B|^{2s} \left|B^*\right|^{4t} |B|^{2s}\right] - c^2\left[|B|^{2t} \left|B^*\right|^{4s} |B|^{2t}\right]$$

$$\leq |B|^{2s} |B|^{4t} |B|^{2s} - c^2 |B|^{2t} |B|^{4s} |B|^{2t}$$

$$\leq |B|^{4(s+t)} - c^2 |B|^{4(s+t)}$$

$\leq \left(1-c^2\right)|B|^{4(s+t)}$

$\leq 0, c > 0$

$\Rightarrow$B(s,t) is quasiposinormal.

## 5.Results:

1. $B^{*^2}(s,t) = |B|^{2t} \cup^* |B|^{2s}$

2. $B^2(s,t) = |B|^{2s} \cup |B|^{2t}$

3 $\left|B^2(s,t)\right| = \left[ B^{*^2}(s,t) B^2(s,t) \right]^{\frac{1}{2}}$

4. $|B(s,t)|^2 = B^*(s,t)B(s,t)$

## 6.Class A operators

**Theorem 6.1:**

If B is semi-hyponormal operator in a Hilbert space H,then B(s,t) is of class A

**Proof:**

An operator B is in class A if

$\left(B^* |B|^2 B\right)^{\frac{1}{2}} \geq B^* B$

(or)

$$\left(B^*|B|^2 B\right) \geq \left(B^*B\right)^2$$

$$\Rightarrow \left[B^*(s,t)|B(s,t)|^2 B(s,t)\right] \geq \left[B^*(s,t)B(s,t)\right]^2$$

$$LHS = B^*(s,t)|B(s,t)|^2 B(s,t)$$

$$= B^*(s,t)B^*(s,t)B(s,t)B(s,t)$$

$$= B^{*^2}(s,t)B^2(s,t)$$

$$= |B|^{2t} \cup^* |B|^{2s} |B|^{2s} \cup |B|^{2t}$$

$$= |B|^{2t} \cup^* |B|^{4s} \cup |B|^{2t}$$

$$= |B|^{2t} |B^*|^{4s} |B|^{2t}$$

$$\geq |B^*|^{2t} |B^*|^{4s} |B^*|^{2t}$$

$$\geq |B^*|^{4(s+t)}$$

$$\text{RHS} = \left[B^*(s,t)B(s,t)\right]^2$$

$$= \left[|B|^t \cup^* |B|^{2s} \cup |B|^t\right]^2$$

$$= |B|^{2t} \cup^* |B|^{4s} \cup |B|^{2t}$$

$$= |B|^{2t} |B^*|^{4s} |B|^{2t}$$

$$\geq \left|B^*\right|^{2t} \left|B^*\right|^{4s} \left|B^*\right|^{2t}$$

$$\geq \left|B^*\right|^{4(s+t)}$$

LHS=RHS

$\Rightarrow$ B(s,t) is in class A.

**Theorem 6.2:**

If B is semi-hyponormal operator in a Hilbert space H, then B(s, t) is in quasiclass A.

**Proof:**

An operator B is in class A if

$$B^*\left(\left|B^2\right| - \left|B\right|^2\right) B \geq 0$$

$$B^*\left|B^2\right| B - B^*\left|B\right|^2 B \geq 0$$

$$B^*(s,t)\left|B^2(s,t)\right| B(s,t) - B^*(s,t)\left|B(s,t)\right|^2 B(s,t) \geq 0$$

$$B^*(s,t)\left|B^2(s,t)\right| B(s,t) \geq B^*(s,t)\left|B(s,t)\right|^2 B(s,t)$$

LHS= $B^*(s,t)\left|B^2(s,t)\right| B(s,t)$

$$= \left|B\right|^t \cup^* \left|B\right|^s \left[B^{*^2}(s,t) B^2(s,t)\right]^{\frac{1}{2}} \left|B\right|^s \cup \left|B\right|^t$$

$$= \left|B\right|^t \cup^* \left|B\right|^s \left[B^*(s,t) B(s,t)\right] \left|B\right|^s \cup \left|B\right|^t$$

$$=|B|^t U^* |B|^s \left[ |B|^t U^* |B|^s |B|^s U |B|^t \right] |B|^s U |B|^t$$

$$=|B|^t U^* |B|^s \left[ |B|^t U^* |B|^{2s} U |B|^t \right] |B|^s U |B|^t$$

$$=|T|^t U^* |T|^{2s} U |T|^t \left[ |T|^t |T^*|^{2s} |T|^t \right]$$

$$=|B|^t |B^*|^{2s} |B|^t \left[ |B|^t |B^*|^{2s} |B|^t \right]$$

$$=|B|^t |B^*|^{2s} |B|^{2t} |B^*|^{2s} |B|^t$$

$$=|B|^{2t} |B^*|^{2s} |B|^{2t} |B^*|^{2s}$$

$$\geq |B^*|^{2t} |B^*|^{2s} |B^*|^{2t} |B^*|^{2s}$$

$$\geq |B^*|^{4(s+t)}$$

RHS= $B^*(s,t)|B(s,t)|^2 B(s,t)$

$\geq |B^*|^{4(s+t)}$ (by theorem 4.1)

$\Rightarrow$ B(s,t) belongs to quasiclass A.

**Theorem 6.3:**

If B is semi-hyponormal operator in a Hilbert space H, then B(s,t) is in m-quasiclass A.

**Proof :**

An operator B in B(H) is in m-quasiclass A if

LHS= $B^{*^m}\left(\left|B^2\right|-\left|B\right|^2\right)B^m \geq 0$

$= B^{*^m}(s,t)\left|B^2(s,t)\right|B^m(s,t) - B^{*^m}(s,t)\left|B(s,t)\right|^2 B^m(s,t) \geq 0$

$= B^{*^m}(s,t)\left|B^2(s,t)\right|B^m(s,t) - B^{*^m}(s,t)\left|B(s,t)\right|^2 B^m(s,t)$

$= B^{*^m}(s,t)\left[B^*(s,t)B(s,t)\right]B^m(s,t) - B^{*^m}(s,t)\left[B^*(s,t)B(s,t)\right]B^m(s,t)$

$= B^{*^{(m+1)}}(s,t)B^{m+1}(s,t) - B^{*^{(m+1)}}(s,t)B^{m+1}(s,t)$

=0

$\geq 0$

$\Rightarrow$ B(s,t) belongs to m quasiclass A

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